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Transformations in Special Relativity

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By using the principle of relativity alone (no assumptions about signals or light) it is shown that a relativisitic group of linear transformations of a spacetime plane is, if infinite, either Galilean, Lorentzian or rotational. The largest such finite group is a Klein 4-group, generated by space-reversal and time-reversal. In the infinite case an invariant of the group, denoted c, appears. When c is real, nonzero, noninfinite, then the group is a Lorentz group and c is identified with the speed of light. Lorentz transformations are represented through an algebra $\mathbb{D}$ of *iterants* that provides a link among Clifford algebras, the Pauli algebra and Herman Bondi's K-calculus.

1. INTRODUCTION

The purpose of this paper is to prove a theorem about groups of transformations of R^2 that obey the principle of special relativity. The theorem is as follows:

Theorem A. Let G be a group of linear transformations of the real plane R^2 to itself. Let $\sigma: R^2 \to R^2$ be the map defined by the formula

$$\sigma(a, b) = (a, -b)$$

Suppose that every element $T \in G$ satisfies the relation $(T \circ \sigma)^2 = I$, where $\circ$ denotes composition and I is the identity transformation. (We say G is *maximal* if it satisfies the hypothesis $(T \circ \sigma)^2 = I$, and is not contained in any larger group that satisfies this hypothesis.) Then:

(i) If G is maximal and of infinite order then there is a constant c that is an invariant of G (c may be computed from any element of G that is not equal to plus or minus the identity) such that:

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1. If $c=0$ or $c=\infty$ then G is isomorphic to a group of Galilean transformations.
2. If c is real and neither zero nor infinite, then G is isomorphic to the Lorentz group for one space dimension and one time dimension.
3. If c is imaginary, then G is isomorphic to $SO(2)$, the group of rotations of the plane.

These three cases exhaust the possibilities for G of infinite order.

(ii) If G is maximal and of finite order, then G is isomorphic to the Klein four-group $Z_2 \times Z_2$.

We will explain how the hypothesis $(T \circ \sigma)^2 = I$ corresponds to the principle of relativity in Section 2. The theorem is proved in Section 4. Examples and relationships with other formalisms are discussed in Sections 3, 5, 6, and 7.

Theorem A is closely related to the results of Edmund DiMarzio (1977).

In the case of a real constant, when G is a Lorentz group, this constant corresponds to the speed of light (in the usual physical interpretation). It is remarkable that these physical features appear inevitably in the abstract framework. Our version of the relativity principle makes no assumption about limiting velocity, or about the speed of light.

Along with Theorem A, this paper also discusses a particular mode of representation for Lorentz transformations. This mode uses an algebra $\mathbb{D}$ that is analogous to the complex numbers. This algebra consists in numbers of the form $a+ib$ with a and b real.

Multiplication in $\mathbb{D}$ is denoted by $*$, and $i * i = +1$. Thus $(a+ib) * (c+id) = (ac+bd) + i(ad+bc)$. This structure is particularly well suited to special relativity in the case where the speed of light is equal to one. In Section 3 we show how the Lorentz transformation is given by the formula

$$t' + ix' = [(1+iv)/(1-v^2)^{1/2}] * (t+ix)$$

when the transformation is between inertial frames with relative velocity v.

In Section 5 we reexpress $\mathbb{D}$ in what I call iterant coordinates $[A, B]$. Here

$$[A, B] = ((A+B)/2) + i((A-B)/2) \text{ and } [a+b, a-b] = a+ib$$

Thus $i=[+1, -1]$ in the iterant coordinates, and one can think of i and $-i(=[-1, +1])$ as representing two views of the process

$$-+-+-+-+-+-+-+-+-+-+-$$

(This process can be seen as a repetition of $[+, -]$ or as a repetition of $[-, +]$.)

In the iterant coordinates the Lorentz transformations have the particularly simple form

$$T[A, B] = [K^{-1}A, KB]$$

The physical interpretation of these coordinates comes through the identity

$$t+ix=[t+x,\ t-x]$$

In Section 6 we show how the pair $[t+x, t-x]$ can be regarded as two time measurements by an observer: $(t-x)$ is the time of emission of a light flash. $(t+x)$ is the reception time of a reflection of this flash from an event E. It follows that the event has space-time coordinates (t, x) for this observer. The iterant formulation provides a link with the K calculus of Herman Bondi (1964).

Finally, in Section 7 we combine the dual numbers $\mathbb{D}$ with the complex numbers $\mathbb{C}$ to form $M=\mathbb{D}\times\mathbb{C}$, a four-dimensional space-time. This leads directly to the Hermitian formalism, the Pauli matrices, and to quaternionic transformations in special relativity.

In regard to Theorem A, it is worth mentioning that in the case of the Klein 4-group, G is generated by space reflection $[\sigma(t, x)=(t, -x)]$ and time-reflection $[\sigma'(t, x)=(-t, x)]$. Except for this finite case, time is constrained by the principle of relativity to flow forward.

2. THE PRINCIPLE OF RELATIVITY

It is well known that space-time coordinates for inertial frames are related by a linear transformation. Furthermore, spatial coordinates perpendicular to the direction of motion are left invariant. Consequently, it suffices to consider linear transformations of R,

$$(t', x')=T(t, x)$$

where x corresponds to the direction of motion, and t corresponds to the (direction of) time.

Call the coordinates (t, x) and (t', x') compatible if the positive direction for x is also the positive direction for x'. See Figure 1.

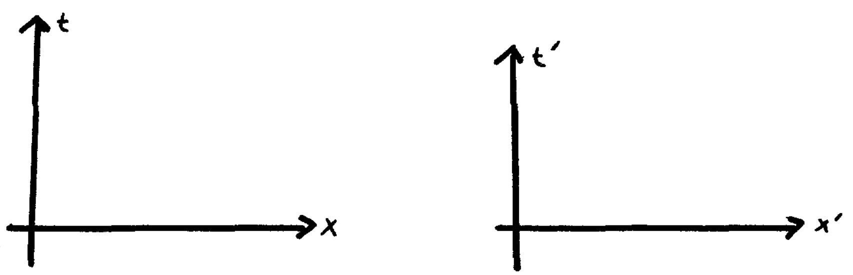

Fig. 1. Compatible coordinates.

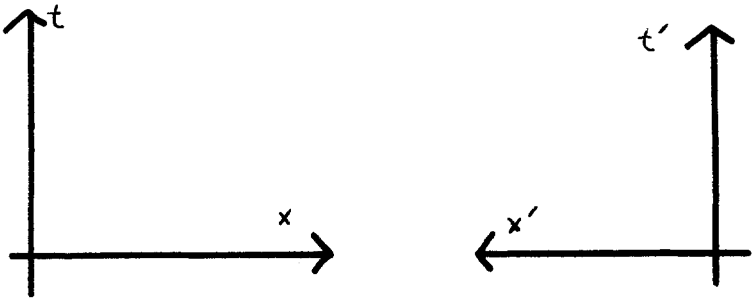

Fig. 2. Mirrored coordinates.

It may also happen that the positive direction for x is the negative direction for x'. In this case, call the coordinates mirrored. See Figure 2. The following is then *a mathematical form of the principle of special relativity*:

(0) If T is a transformation of inertial frames, then T is invertible, and its inverse is also a transformation of these frames.

(1) If $0, 0', 0''$ are three compatible frames and $T: 0 \to 0'$, $T': 0' \to 0''$, $T'': 0 \to 0''$ are the transformations relating them, then $T'' = T' \circ T$.

(2) If 0 and $0'$ are mirrored frames and $S: 0 \to 0'$ is the corresponding transformation, then the transformation from $0'$ to 0 is also given by S, and [by (0)] $S \circ S = I$.

Statement number (2) embodies the simplest instance of the intuitive relativity principle. The mirrored frames are symmetrical with respect to each other. Hence they must be related by the same transformation.

It is worth noting that while we are allowed to reflect the space coordinate via $\sigma(t, x) = (t, -x)$, we have no such freedom to reverse the direction of time.

Since $\sigma: R^2 \to R^2$ interchanges compatible and mirrored coordinates, it follows from (0), (1), and (2) that $(T \circ \sigma)^2 = I$ whenever $T: R^2 \to R^2$ is a relativistic transformation of compatible frames.

Hence we conclude that the set G of all relativistic transformations of compatible frames (all compatible with one another) forms a group under composition. And $(T \circ \sigma)^2 = I$ for every T in G.

This completes the explanation of our choice of hypotheses for Theorem A.

3. PRELUDE TO THEOREM A

In this section we show how to deduce the Lorentz transformation in a special context. In Section 4 this argument will be generalized to become the proof of Theorem A.

Let $\mathbb{D} = \{t + ix | t, x \in R\}$. Here i is a symbol satisfying the identity $i * i = +1$. (Multiplication in $\mathbb{D}$ is denoted by $*$.) $\mathbb{D}$ will be referred to as the set of dual numbers. The dual numbers are formally similar to the complex numbers. Thus multiplication is given by the formula $(a+ib) * (c+id) = (ac+bd) + i(ad+bc)$. Conjugation is defined by the formula

$$\sigma(a+ib) = \overline{(a+ib)} = a - ib$$

Note that $a^2 - b^2 = (a+ib) * (a-ib)$. Thus $\mathbb{D}$ embodies the hyperbolic metric of Einstein's special relativity.

The dual numbers may also be identified with a subalgebra of the Pauli algebra. This connection will be discussed in Section 7.

In this section we derive Lorentz transformations by adopting the following.

Assumption $\mathbb{D}$. Suppose that G is a group of transformations of $\mathbb{D}$ such that (1) for each T in G there is an element w of $\mathbb{D}$ such that $T(z) = w * z$ for all z in $\mathbb{D}$ and (2) $(T \circ \sigma)^2 = I$ for each T in G.

Proposition 3.1. Assumption $\mathbb{D}$ implies that G is the Lorentz group (light speed normalized to 1).

Proof. Let $T \in G$ and $w \in \mathbb{D}$ so that $T(z) = w * z$ for all $z \in \mathbb{D}$. The condition $(T \circ \sigma)^2 = I$ implies that $w * \overline{(w * \bar{z})} = z$ for all z in $\mathbb{D}$. Since $\overline{X * Y} = \bar{X} * \bar{Y}$ and $\bar{\bar{X}} = X$ for all X, Y in $\mathbb{D}$, we have $(w * \bar{w}) * z = z$ for all z in $\mathbb{D}$. This implies that $w * \bar{w} = 1$.

To obtain the specific form of T, let $w = a + ib$. Then $w * \bar{w} = 1$ implies that $aa - bb = 1$. Hence $w = (1 + iv)/(1 - v^2)^{1/2}$ where $v = b/a$. Note that v has the dimensions of velocity (if we interpret b as position and a as time). Then

$$T(t+ix) = [(1+iv)/(1-v^2)^{1/2}] * (t+ix)$$

$$T(t+ix) = [(t+vx)/(1-v^2)^{1/2}] + i[(x+vt)/(1-v^2)^{1/2}]$$

Thus T is a Lorentz transformation for frames with relative velocity v, and light speed equal to 1. This completes the proof of the proposition. ■

Remark. In this derivation we did not assume that T left the metric $z * \bar{z} = (t+ix) * (t-ix) = t^2 - x^2$ invariant. Nevertheless, this invariance is a consequence of the restriction $w * \bar{w} = 1$. That is,

$$T(z) * \overline{T(z)} = (w * z) * \overline{(w * z)}$$

$$(w * z) * \overline{(w * z)} = (w * \bar{w}) * (z * \bar{z})$$

$$\therefore\ T(z) * \overline{T(z)} = z * \bar{z}$$

Hence $t'^2 - x'^2 = t^2 - x^2$.

The Lorentz transformations are a consequence of the use of the dual numbers. If $\mathbb{D}$ is replaced by the complex numbers $\mathbb{C}$ (where $ii=-1$), then $T(z)=wz$ with $w\bar{w}=1$ implies that T is a rotation of the plane about the origin. This corresponds to part 3 of Theorem A.

4. PROOF OF THEOREM A

We now assume that G is a group of 2×2 real matrices. Let

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma$$

so that

$$\sigma\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ -x \end{pmatrix}$$

By hypothesis, $T\in G$ implies that $(T\circ\sigma)^2=I$ where I is the identity matrix.

For any 2×2 matrix

$$M=\begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

the condition $MM=I$ is equivalent to the condition $M=M^{-1}$. This, in turn, is equivalent to

$$\begin{pmatrix} a & b \\ d & c \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} c & -b \\ -d & a \end{pmatrix}$$

where $\Delta = ac-bd$ (the determinant of M). This forces $\Delta=\pm 1$.

If $\Delta=+1$ then

$$\begin{pmatrix} a & b \\ d & c \end{pmatrix} = \begin{pmatrix} c & -b \\ -d & a \end{pmatrix}$$

hence $a=c$, $b=d=0$, and $aa=ac=+1$. Thus $M=\pm I$.

If $M=(T\sigma)$ has determinant equal to -1, then a similar calculation shows that T (not M but T) has the form

$$T=\begin{pmatrix} a & b \\ d & a \end{pmatrix}$$

with $aa-bd=1$. Thus we have the following.

Lemma 4.1. Let T be an invertible 2×2 real matrix with $(T\sigma)^2=I$ where

$$\sigma=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad\text{and}\quad \sigma'=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then (1) $\mathrm{Det}(T)=-1$ implies that T is σ or σ^{-1}; (2) $\mathrm{Det}(T)=+1$ implies that

$$T=\begin{pmatrix} a & b \\ d & a \end{pmatrix}$$

where $aa-bd=1$.

Returning to the proof of Theorem A, if G contains σ and G also contains an element T with determinant equal to $+1$, then $T\sigma$ belongs to G. Hence $[(T\sigma)\sigma]^2=1$. Thus $TT=1$. Hence T is ± 1.

Using a similar argument with σ', we obtain the following.

Lemma 4.2. Let G be a group of 2×2 matrices such that $(T\sigma)^2=I$ for all T in G. (σ and σ' are as above.) If σ or σ' belongs to G, then G is isomorphic to one of the following groups:

$$\{1,\sigma\}\cong\mathbb{Z}_2$$

$$\{1,\sigma'\}\cong\mathbb{Z}_2$$

$$\{1,-1,\sigma,\sigma'\}\cong\mathbb{Z}_2\times\mathbb{Z}_2$$

(The latter is referred to as the Klein 4-group.)

Having classified those groups containing σ or σ', we now assume (using 4.1) that all elements of G have the form

$$T=\begin{pmatrix} a & b \\ d & a \end{pmatrix}$$

with $aa-bd=+1$.

Suppose T' is also in G.

$$T'=\begin{pmatrix} a' & b' \\ d' & a' \end{pmatrix}$$

Then TT' belongs to G, and

$$TT'=\begin{pmatrix} a & b \\ d & a \end{pmatrix}\begin{pmatrix} a' & b' \\ d' & a' \end{pmatrix}=\begin{pmatrix} aa'+bd' & ab'+ba' \\ da'+ad' & db'+aa' \end{pmatrix}$$

must also have equal elements on the main diagonal. Therefore $aa'+bd'=db'+aa'$. Hence $bd'=db'$. *Therefore* $d/b=d'/b'$.

It may happen that $G=\pm I$, in which case these formal fractions contain no information. Otherwise, the matrices each have at least one nonzero off-diagonal term. Then $d/b=d'/b'$ is meaningful, taking real values that include 0 and ∞.

By using $c=(d/b)^{1/2}$ we can rewrite T into the form of a Lorentz transformation. The rest of the argument follows by specializing the values of c. The translation:

$$T=\begin{pmatrix} a & b \\ d & a \end{pmatrix}, \qquad a^2-bd=1$$

Let $c=(d/b)^{1/2}$ and $v=d/a$.

$$\Rightarrow T=\frac{1}{(1-v^2/c^2)^{1/2}}\begin{pmatrix} 1 & v/c^2 \\ v & 1 \end{pmatrix}$$

This looks like a Lorentz transformation, and it is a Lorentz transformation when $0<c<\infty$. If $c=\infty$ then

$$T=\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$$

hence $t'=t$ and $x'=vt+x$. This is a Galilean transformation. If $c=0$, then $d=0$, $b\neq 0$ and again we obtain a transformation of Galilean type. Hence the group of transformations with this invariant is isomorphic to the Galilean group. The isomorphism interchanges space and time coordinates.

If $-\infty<c^2<0$, then T preserves the form $c^2t^2+x^2$ and hence is a rotation of the $(|c|t, x)$ plane.

These observations and lemmas combine to give the proof of Theorem A.

Remark. When $c=1$ we have $d=b$, hence

$$T=\begin{pmatrix} a & b \\ b & a \end{pmatrix}=a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The algebra of these matrices (without restriction on a and b) is isomorphic to the dual numbers $\mathbb{D}$ introduced in Section 3. In this form, each point in space-time is represented by a matrix

$$\begin{pmatrix} t & x \\ x & t \end{pmatrix}$$

The Lorentz transformations are the subalgebra of matrices satisfying $t^2-x^2=1$.

5. REMARKS ON THE DUAL NUMBERS

We have seen in Sections 3 and 4 that the dual numbers $\mathbb{D}=\{a+ib \mid i*i=+1\}$ bear a close relationship with special relativity. Lorentz transformations are represented by $T(z)=w*z$, where $w=$

$(1+iv)/(1+v^2)^{1/2}$. In this section we make some remarks about the pattern of this number system.

Consider the following linear pattern:

... *ABABABABABABABABABABABABABABABABABABABABABAB*...

This can be seen as an endless repetition of *AB* or as an endless repetition of *BA*. Accordingly, we choose the notations $[A, B]$ and $[B, A]$ to represent the two views of the pattern. Call $[A, B]$ the conjugate of $[B, A]$. And write $\overline{[A, B]} = [B, A]$.

There is an algebra for combining these patterns:

$$[A, B] + [C, D] = [A + C, B + D]$$

$$[A, B] * [C, D] = [AC, BD]$$

If we take the elements of these patterns to be real numbers, then the iterant algebra is isomorphic to the dual numbers $\mathbb{D}$. The isomorphism is given by the correspondence $a + ib = [a + b, a - b]$. Thus $1 = [1, 1]$ and $i = [1, -1]$. See Comfort (1984), Kauffman (1980), Kauffman and Varela (1980), and Kauffman (to appear) for further comments.

This context for the iterant algebra invokes the concept of multiple viewpoint. The iterant $[A, B]$, and its conjugate $[B, A]$ are mathematical analogs of two mutually exclusive perceptions of a single form. In this model, form is represented by the periodic process ... *ABABABABABA* .... Just so, for an observer, a form or object with spatial and temporal extent is seen as an (apparently) periodic series of observations—all views of one whole.

In the next section we shall see that, for an observer, an event in space-time can be idealized as two time-measurements: one of signal sent, one of signal received. In this form an object (process) becomes a series of such observations in the form: ... SEND, RECEIVE, SEND, RECEIVE, SEND, RECEIVE, ... The crucial choice of what is regarded as sent, and what is regarded as received is determined largely by context. This context includes agreement throughout the given community of observers. Iterant algebra abstracts only the simplest elements of this complex process.

6. DUAL NUMBERS AND THE *K* CALCULUS

The last section remarked on the mathematical pattern of the dual numbers, and introduced the iterant coordinates:

$$[A, B] = [(A + B)/2] + i[(A - B)/2]$$

$$[a + b, a - b] = a + ib$$

An event (t, x) is represented in $\mathbb{D}$ as $t+ix=[t+x, t-x]$. One physical interpretation of the iterant coordinates $[t+x, t-x]$ is given by Herman Bondi (1964) in the context of his K calculus. Here is a direct quote [Bondi (1964), pp. 116–118]:

> Let Alfred use coordinates t, x and Brian coordinates t', x', so that Alfred is $x=0$, Brian is $x'=0$, and at the meeting of Alfred and Brian $t=t'=0$. Consider an event which, seen by Alfred, is beyond Brian [Figure 3; Bondi's Figure 23]. Alfred emits a radar pulse at time $t-x$ and receives it back at time $t+x$ so that he assigns coordinates t, x to the event. Similarly, Brian emits a pulse at $t'-x'$ and gets it back at $t'+x'$.
>
> But in fact Brian emits his pulse as Alfred's pulse passes him and receives it as the returning pulse to Alfred passes him. Hence $t'-x'=K(t-x)$ and $t+x=K(t'+x')$.

In Bondi's terminology the observers Alfred and Brian are separating at constant velocity. The lines labeled Alfred and Brian in Figure 3 represent their respective time coordinates (world lines) in two-dimensional space-

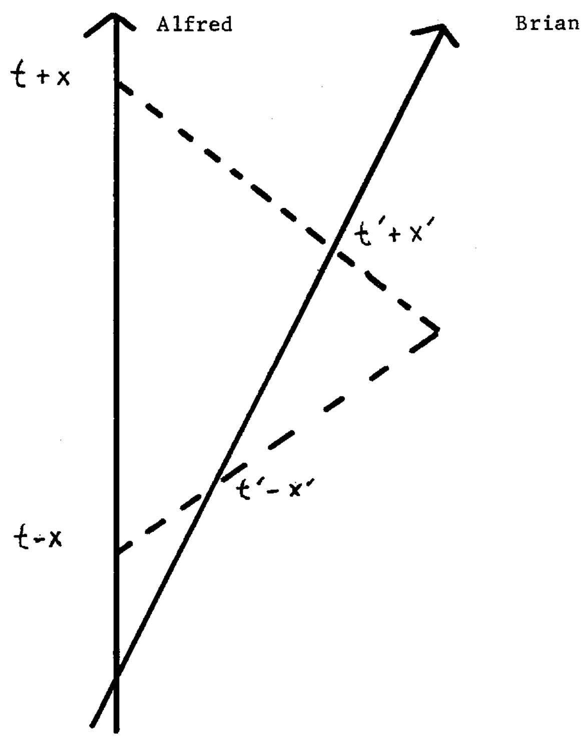

Fig. 3. The Lorentz transformation.

time. The constant K is the ratio $K=$(interval of reception)/(interval of transmission). Thus, if Alfred sends a pulse at time $(t-x)$ on his world line, and the pulse is received at time $(t'-x')$ on Brian's world line then $K=(t'-x')/(t-x)$.

The rest of Bondi's argument uses the geometry of the figure plus the consequence (of the principle of relativity and constancy of light speed) that K remains the same when the roles of Alfred and Brian are interchanged. In this way the Lorentz transformation takes the form

$$[t'+x', t'-x'] = [K^{-1}(t+x), K(t-x)] \qquad \text{or}$$

$$T[A, B] = [K^{-1}A, KB]$$

This is exactly the form of the Lorentz transformation in iterant coordinates.

It is useful to derive the relation between K and the velocity v (v is the relative velocity of the two frames). Since the Lorentz transformation is represented by $(1-iv)/(1-v^2)^{1/2}$ for this geometry, we have $(1-iv)/(1-v^2)^{1/2} = [K^{-1}, K]$. Hence $(1-iv)/(1-v^2)^{1/2} = [(K^{-1}+K)/2] + i[(K^{-1}-K)/2]$. Therefore $v=(K-K^{-1})/(K+K^{-1}) = (K^2-1)/(K^2+1)$. This relationship can be derived directly in the K calculus by considering another thought experiment involving transmission and reception (Bondi, 1964, p. 103).

These remarks exhibit the physical meaning of the iterant coordinates. From the point of view of a given observer, an event is indexed by a pair $[A, B]$, where B is the time of emission of a signal, and A is the time of reception of another (correlated) signal. It is through the patterning of such pairs that we create descriptions of the world of events.

7. COMBINING DUAL NUMBERS AND COMPLEX NUMBERS

Four-dimensional space-time can be obtained by combining the complex numbers $\mathbb{C}$ with the dual numbers $\mathbb{D}$. We can also go to four-space directly by remarking that in the formalism $t+ix$ the i represents a vector direction in three-dimensional space. As this leads directly to the notion of a Clifford algebra, we examine this point of view first.

See Figure 4. Here i is written as a linear combination of basis vectors σ_1, σ_2, σ_3 for Euclidean three-space. The basis vectors are orthogonal and of unit length. It is assumed that i has unit length, hence $x^2+y^2+z^2=1$.

In the dual numbers, and in order to represent Lorentz transformations we assumed that $i*i=+1$. If we assume that this algebraic structure can

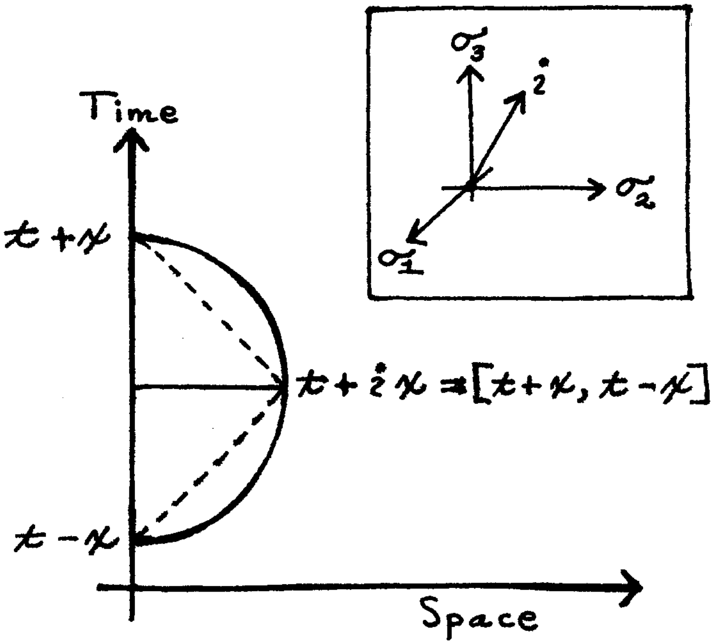

Fig. 4. Lorentz transform $\{[t'+x', t'-x'] = [K^{-1}, K] * [t+x, t-x]$; $[A, B] * [C, D] = [AC, BD]\}$; $i = [1, -1] \Rightarrow i * i = [1, 1] = 1$; $i = x\sigma_1 + y\sigma_2 + z\sigma_3 \Rightarrow$ Clifford algebra

$$\left\{\begin{array}{l} \sigma_1 * \sigma_1 = \sigma_2 * \sigma_2 = \sigma_3 * \sigma_3 = 1 \\ \sigma_1 * \sigma_2 = -\sigma_2 * \sigma_1 \\ \sigma_1 * \sigma_3 = -\sigma_3 * \sigma_1 \\ \sigma_2 * \sigma_3 = -\sigma_3 * \sigma_2 \end{array}\right\}$$

be extended to all the vector directions in three-space, then it follows that

$$i^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = +1 \qquad (\mathbf{x}^2 = \mathbf{x} * \mathbf{x})$$

$$i = x\sigma_1 + y\sigma_2 + z\sigma_3, \qquad x^2 + y^2 + z^2 = 1$$

$$1 = i * i$$

$$1 = \begin{pmatrix} x^2 + y^2 + z^2 + (\sigma_1 * \sigma_2 + \sigma_2 * \sigma_1)xy \\ + (\sigma_1 * \sigma_3 + \sigma_3 * \sigma_1)xz \\ + (\sigma_2 * \sigma_3 + \sigma_3 * \sigma_2)yz \end{pmatrix}$$

$$\therefore \ \sigma_i * \sigma_j = -\sigma_j * \sigma_i, \qquad i \neq j$$

This means that the unit directions must form a Clifford algebra. The simplest model is the Pauli algebra where

$$\sigma_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}0 & (-1)^{1/2}\\ -(-1)^{1/2} & 0\end{pmatrix}$$

$$E=t+x\sigma_1+y\sigma_2+z\sigma_3=\begin{pmatrix}t+x & y+(-1)^{1/2}z\\ y-(-1)^{1/2}z & t-x\end{pmatrix}$$

In this way an event is represented by a Hermitian matrix H (above). It is curious to note that this matrix appears as a display of an element $t+ix\in\mathbb{D}$ where $t+ix=[t+x,\ t-x]$ is displayed on the main diagonal in iterant form, while $y+(-1)^{1/2}z$ is displayed on the minor diagonal in tandem with its complex conjugate.

In this way we obtain a mapping of $\mathbb{D}\times\mathbb{C}$ into conventional space-time so that

$$M=\mathbb{D}\times\mathbb{C}\longrightarrow \text{space-time}$$

$$(t+ix,\ y+(-1)^{1/2}z)\longmapsto\begin{pmatrix}t+x & y+(-1)^{1/2}z\\ y-(-1)^{1/2}z & t-x\end{pmatrix}$$

By defining $s\colon M\to R$ via the formula $s(Z,W)=Z*\bar{Z}-W\bar{W}$, we retrieve the space-time interval:

$$s[x+it,\ y+(-1)^{1/2}z]=t^2-x^2-y^2-z^2$$

(Note that this corresponds to the determinant of the Hermitian matrix.)

A generalization of the argument in Section 3 shows how to derive the form of the Lorentz transformation in the Pauli algebra:

Let $T(E)=A*E*B$ where $E=t+x\sigma_1+y\sigma_2+z\sigma_3$ is an event, and A and B are given elements of the Pauli algebra. In order for T to be a Lorentz transformation it is necessary that $(T\sigma)^2=I$ where σ denotes conjugation. Hence

$$A(\overline{A\bar{E}B})B=E,\qquad \forall_E$$

$$A(\bar{B}\bar{\bar{E}}\bar{A})B=E,\qquad \forall_E$$

$$(A\bar{B})E(\bar{A}B)=E,\qquad \forall_E$$

Therefore $A\bar{B}=1$, and by normalizing the radius we can take $A=B$ with $A\bar{A}=1$. Thus $T(E)=AEA$. (Dropping the use of $*$.)

In the Hermitian formalism this corresponds to the fact that $SL(2,C)$ double covers the Lorentz group. Quaternionic formalism is obtained by writing an event in the form

$$e=(-1)^{1/2}t+x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

where $\mathbf{i}=(-1)^{1/2}\sigma_1, \mathbf{j}=(-1)^{1/2}\sigma_2, \mathbf{k}=(-1)^{1/2}\sigma_3$.

Here **i**, **j**, and **k** generate the quaternions. In a sequel to this paper we shall examine the relationships among these ideas and the mathematics of twistor theory (Penrose, 1977). Complexification of the Hermitian formalism leads into the geometry of twistor space.

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